Mathematical Explanation of Physical Phenomena: A Step Toward Pluralism?

Abstract

A traditional trend toward scientific explanation aimed to provide a single account of the nature of explanation. On the other hand, a recent attitude toward mathematical explanations of physical phenomena is to go 'pluralist' and consider that what makes something a good explanation can vary from case to case, i.e. there are different kinds of explanation which cannot be captured through a single model. In this paper I will show how two classical theories of scientific explanation (Van Fraassen's pragmatic model and Kitcher's unification model) have difficulties in accounting for a case of mathematical explanation of physical phenomena recognized as such in scientific practice. Furthermore, I will argue that the result of this testing does suggest a new picture of mathematical explanations of physical phenomena. According to this new perspective, the pluralist attitude should be considered as the most promising way to the investigation of mathematical explanation of physical phenomena.

Keywords: Mathematical Explanation, Visual Reasoning, Asymptotic Reasoning, Hénon-Heiles, Unification, Why-Questions

1 Introduction

Although there is far from general consensus that mathematics does play an explanatory role in physics (Daly and Langford, 2009), the existence of mathematical explanations of physical phenomena is widely recognized in the literature (Steiner, 1978; Batterman, 2002; Pincock, 2007; Lyon and Colyvan, 2008). Moreover, it is beyond question that the topic of mathematical explanation of physical phenomena (henceforth MEPP) has received a particular attention among contemporary philosophers of science (Mancosu, 2008). This is particularly evident if we consider how MEPP, together with the related notion of 'explanatory power of mathematics', appear as a crucial ingredient in various and distinct philosophical topics (indispensability argument, applicability of mathematics, scientific understanding, mathematical modelling and idealization, etc.).

It is often observed that the leading contemporary theories of scientific explanation are in trouble when faced with MEPP (Baker, 2005; Batterman, 2002; Mancosu, 2008). In fact, in some cases these theories left apart mathematical explanations and they did not accept mathematical statements within their structure. This is the case of Hempel's Deductive-Nomological model (Hempel and Oppenheim, 1948), in which the explanans must have empirical content (they must capable, at least in principle, of being tested by means of experiments and observations). In other cases those accounts had the ambition to cover MEPP as well but their structure was found to be insufficient for the treatment of specific cases of MEPP. This is what happened, for instance, with Kitcher's unification model (Kitcher, 1989)[1]. Finally, if mathematical objects are acausal, MEPP represent counterexamples to causal models of scientific explanation, such as Salmon's (Salmon, 1989), which consider explanation in the natural sciences as esssentially causal. By assuming that mathematical objects do not play any essential role in the explanation provided, causal models miss MEPP simply because they rule out the possibility to have such a kind of explanations. Of course, all the previous considerations point to some substantial impediments the major theories of scientific explanation have when confronted with MEPP. Despite the great interest in the linkage scientific explanation-MEPP, however, an extensive discussion of models of scientific explanation in the context of MEPP has not been offered and work is just beginning.

The traditional tendency toward scientific explanation has been to capture the nature of explanation by providing a single model, i.e. a model whose aim was to define explanation *simpliciter*. Call this approach the 'winner-take-all' (WTA) approach to explanation. Although the WTA approach has been extremely influential in the debate on scientific explanation, a contemporary attitude toward MEPP is to go 'pluralist' and consider that MEPP are heterogeneous and we can capture a specific sense (or species) of MEPP for a specific situation (or class of situations) in science (Pincock, 2007; Batterman, 2010). What makes something a good MEPP can vary from case to case. That is, there are different kinds of mathematical explanation in science and we do not design a single model able to capture all the instances of MEPP.

Leaving apart ontological questions and mysteries about the applicability of mathematics, i.e. Eugene Wigner's "unreasonable effectiveness of mathematics in the natural sciences" (Wigner, 1960), some authors agree that it is possible to have a better comprehension of MEPP starting from clues which have been proposed in the context of scientific explanation and focusing on particular case-studies (Pincock, 2007; Mancosu, 2008; Batterman, 2010). In this paper I will follow this line. I will consider a particular case study for philosophical considerations. Furthermore, I will show how the testing of two traditional WTA theories of scientific explanation, Van Fraassen's pragmatic model and Kitcher's unification model, on this case study can inform the debate concerning MEPP and suggest a new direction of investigation.

The article is structured as follows. In the next section I will present Kitcher's and Van Fraassen's theories of scientific explanation. In Section 3, I will consider a case which is recognized in contemporary scientific practice as a case of genuine MEPP: the behaviour of Hénon-Heiles systems explained via the phase space formalism. Then, in Section 4, I will assess the two models against this case study. I will show that the two WTA models of explanation have difficulties in accounting for this MEPP, thus contradicting the intuitions coming from the practice of scientists. Finally, in the conclusive section, I will suggest that the result of this testing unveils a picture of MEPP which stands as yet in need of detailed investigation. According to this view, there are different forms of reasoning which are used in MEPP, and the use of these forms of reasoning should be considered as a crucial component of 'what makes something a good explanation'. This perspective, which will be sketched in the final part of my study, is out of tune with the contemporary WTA approach to explanation and it looks at pluralism as the attitude to adopt in the investigation of MEPP.

2 Kitcher and Van Fraassen on explanation

Let me begin by reviewing two major models of explanation we have in our hands: Kitcher's and Van Fraassen's. Together with the accounts, I will report the criticisms which have been raised to these models. A look at these criticisms will be of extreme importance for my discussion.

2.1 Kitcher's Model of MEPP

The concept of unity in physics has a long history. As Klein and Lachieze-Rey have well illustrated in their book *The Quest for Unity: The Adventure of Physics* (1999), the search for unification begins with Greek conceptions of unity and arrives until our day. It is well known how important the role of mathematics in the process of inclusion of separate theories and phenomena into one single framework has been, and continues to be. This is the case, for example, of Maxwell's famous unification of electromagnetism and optics through the Lagrangian formalism. But how does mathematics play this unifying role? And, for what interests us, does this unification have something to do with explanation? If yes, and if mathematics serves as a unifying tool, how should we characterize a theory of explanation in terms of unification? The unification model for explanation, first proposed by Michael Friedman in (Friedman, 1974) and successively modified and extended by Philip Kitcher in (Kitcher, 1981, 1989), represents an effort to find such a theory. Due to its more elaborated structure, Kitcher's model will be the unification account I will refer to.

Kitcher has presented his model in two different works: (Kitcher, 1981) and (Kitcher, 1989). While originally the model was addressed to general scientific explanation, the fact that such a theoretical account could potentially cover mathematical explanations of physical phenomena as well as mathematical explanations within mathematics is an opinion shared by a number of philosophers (Hafner and Mancosu, 2005; Mancosu, 2008 and Tappenden, 2005). Without entering here into the technicalities of the model, I will present the general idea.

Consider a consistent and deductively closed set K of beliefs endorsed by a scientific community at a particular time, and call a systematization Σ of K a set of arguments which derive some members of K from other members of K. Naturally, there could be different ways to derive some members of K from other members of K. However, according to Kitcher, there is exactly one set $E(K)$ of arguments (called the 'explanatory store' over K) which offers the best systematization of K. Observe here that Kitcher explicitly makes an idealization by claiming that $E(K)$ is unique (Kitcher, 1981, p. 512). An explanation is an act performed by a member of a scientific community whose beliefs include K and which draws on an argument from $E(K)$. The fundamental task of a theory of explanation is then to specify the conditions on the explanatory store. By considering unification as the criterion for systematization, Kitcher's theory takes $E(K)$ as the set of derivations that best *unifies* K.

To evaluate the degree of unification of a systematization Σ, Kitcher introduces the notion of 'argument pattern', i.e. a triple consisting of a 'schematic argument', a set of 'filling instructions' and a 'classification' for the schematic argument. A particular derivation, for instance a sequence of sentences and formulas which accord with Newton's laws, instantiates a general argument pattern. The intuitive idea behind unification is that $E(K)$ is the set of derivations that makes the best trade-off between minimizing the number of patterns of derivation employed and maximizing the number of conclusions generated (call this the 'unification criterion').

Although a great number of studies have been carried out on Kitcher's model in the context of general philosophy of science, the account has not been extensively discussed for cases of mathematical explanation within mathematics and MEPP (Mancosu, 2008). A general criticism of Friedman's and Kitcher's unification approach has been put forward by Margaret Morrison, who claims that unification has little if anything to do with explanation. For her explanation and unification are different (and sometimes conflicting) business, and very often they pull in different directions. The point is stated many times throughout Morrison's book *Unifying Scientific Theories* (2000). A discussion of Kitcher's model for the case of mathematical explanations is found in (Tappenden, 2005), and in (Hafner and Mancosu, 2008) the model is tested by taking into consideration a test-case from real algebraic geometry. Both those two criticisms deal with cases of mathematical explanation within mathematics (explanations recognized as such in the practice of mathematicians), and both give primary importance to the intuitions of mathematicians in their discussions. Johannes Hafner and Paolo Mancosu argue that Kitcher's model makes predictions about explanatoriness that go against specific cases in mathematical practice. Their lesson is that Kitcher's model must be supplemented with some qualitative reinforcement in order to account for the intuitions of mathematicians (Hafner and

Mancosu, 2008, p. 233). As Hafner and Mancosu, Tappenden also observes how the existing accounts of unification are more balanced on quantitative restrictions (for instance, the number of patterns employed) and need to be supplemented with some qualitative component. His point is that only such qualitative reinforcements would permit the unificationist to reflect the actual mathematical practice. Moreover, Tappenden observes how such qualitative injections would permit Kitcher to have a unified treatment of explanations in mathematics and in natural sciences (Tappenden, 2005, p. 174).

2.2 Van Fraassen's Account

Bas van Fraassen has proposed his pragmatic theory of scientific explanation in his book *The Scientific Image* (1980). Perhaps the best way to introduce his view on explanation is to quote two passages from Van Fraassen himself:

> There are no explanations in science. How did philosophers come to mislocate explanation among semantic rather than pragmatic relations? (Van Fraassen, 1977, p. 150)

> An explanation is not the same as a proposition, or an argument, or list of propositions. [...] An explanation is an answer to a why question. So, a theory of explanation must be a theory of why-questions. (Van Fraassen, 1980, p. 134)

Thus, according to Van Fraassen, there is no scientific explanation simpliciter, but explanations are relative to the context dependent why-questions they answer. What is requested in order to respond to the question 'Why is it the case that P?' differs from context to context, and the question arises in a context with a certain body of accepted theory plus information. However, as I affirmed in the Introduction to the present study, I am considering Van Fraassen's account among the WTA conception of explanation, i. e. the conception of explanation in which the model is designed in order to capture a general sense of explanation. How do I justify my claim? To understand my point, let me offer a short illustration of his account.

Van Fraassen's approach to the general logic of questions was inspired by Belnap and Steel's book *The Logic of Questions and Answers* (1976), with some additional refinements in order to fit that theory with other studies on scientific explanation. For Van Fraassen a necessary prerequisite for an explanation is that there is a why-question. But what exactly is a why question? In Van Fraassen's model a why-question is a triple $Q = \langle P_k, X, R\rangle$ consisting of:

- a topic P_k
- a constrast class $X = \{P_1, ..., P_k, ...\}$
- a relevance relation R

When we ask 'Why P_k?' we refer to a proposition P_k called the 'topic' of our question (P_k expresses the fact to be explained, i.e. the explanandum). The *contrast-class* of the question is a set of alternatives, that is, a class X of propositions $\{P_1, ..., P_k, ...\}$ which includes the topic P_k. The propositions (or alternatives) P_i belonging to X are propositions expressing possibilities the questioner is willing to consider, including P_k. Finally, a *relevance relation*

R is the "respect-in-which a reason is requested". The relevance relation is used to constrain admissible answers, by specifying what factors will count as explanatorily relevant and thus by distinguishing between different senses of the question. A proposition A is called 'relevant to a why question Q if A bears relation R to the couple $\langle P_k, X\rangle$[2]. Answers to such a question Q differ from non-answers because they have the following form of words: "P_k *in constrast to* (the rest of) X *because* A', where the word 'because' indicates that A is a *reason*. More precisely, the word 'because' guarantees that A is relevant, in this context, to the question, i.e. that it bears relation R to $\langle P_k, X\rangle$. Thus Van Fraassen proposes a definition of the notion of 'direct answer', i.e. what counts as an answer to a why-question:

- B is a *direct answer* to question $Q = \langle P_k, X, R\rangle$ exactly if there is some proposition A such that A bears relation R to $\langle P_k, X\rangle$ and B is the proposition which is true exactly if (P_k; and for all $i \neq k$, not P_i; and A) is true, where $X = \{P_1, ..., P_k, ...\}$. (Van Fraassen, 1980, p. 144)

For simplicity, call A the *core* of answer B (so that the answer can be abbreviated to 'Because A').

Van Fraassen characterizes explanation as an answer to a why-question, where why-questions are essentially contrastive (that is, they are of the form 'Why P_k, rather than some set of alternatives X'?). Furthermore, the why-questions stipulate a relevance relation R, which is the explanatory relation (for example, causation) any answer must bear to the ordered pair $\langle P_k, X\rangle$. Like Hempel, then, Van Fraassen seeks to explicate explanation by providing a formal schema. However, unlike Hempel, the relation proposed in order to capture explanation is not one of premises to conclusion (via a deductive argument), but one of question to answer (via some relevance relation to be provided). Finally, the evaluation of "how much an answer is telling" relies on three different criteria: 1) The fact that A itself is more probable in light of our knowledge K; 2) The probability that A, and thus the answer, favours the topic P_k against the other members of the contrast class relative to background knowledge (favouring criteria); 3) The fact that the answer is made wholly or partially irrelevant by other answers that could be given.

As in the case of Kitcher, it is not possible to discuss Van Fraassen's model in its full complexity. However, a few remarks concerning the criticisms which have been addressed to it will play a very important role for the the considerations to follow.

Note that, although based on the view that explanation is a process of communication, Van Fraassen's theory of explanation still chooses to explicate the concept of explanation as a formal relationship between question and answer, rather than as a communicative relationship between two individuals. However, by favouring an unrestricted relevance relation, his account is open to trivialization. This is the main moral of the criticism proposed by Philip Kitcher and Wesley Salmon in (Kitcher and Salmon, 1987). They illustrate this by showing that *any* true proposition A can be an indispensable part of an explanation of any topic P_k (with respect to a constrast class X that contains P_k and any assortment of false propositions), and, indeed, that it gets highest marks as an explanation of P_k. If Kitcher and Salmon are right, to leave the relevance relation R undefined amounts to leave Van Fraassen's account open to trivialization. On the other hand, to introduce some formal constraint on the relevance relation would amount to fix an objective criterion of explanatoriness, which is something Van Fraassen does not want for his model.

Up to now I have considered Van Fraassen's among the WTA approaches to explanation. However, Van Fraassen does propose a model in which the relevance relation is open (it depends on the context) and therefore it seems that his account perfectly fit into a pluralist perspective on MEPP. Why then I have considered Van Fraassen's account among the WTA models? According to Van Fraassen, explanations *always* come under the form of answers to why-questions. Then, in regarding explanations as answers to why-questions, Van Fraassen does impose a general schema which does not fit with a pluralist view on explanation[3]. This is why I consider his model as a WTA approach. To consider Van Fraassen's model as a WTA approach is in tune with the moral of the last criticism to I am going to present, namely that proposed by David Sandborg (1998).

Sandborg's analysis concerns the possibility for the pragmatic account to deal with mathematical explanations within mathematics (under the form of proofs). He presents an example of proof from George Polya's book *Patters of Plausible Inference* (Polya, 1968, p. 147) and concludes that a why-question approach does not account for the conceptual resources introduced by the mathematician Polya in his explanation of a particular theorem (Polya's introduction of a particular sequence in what he considers the explanatory proof). According to Sandborg's analysis, a why-question approach misses then an important aspect of the context in which mathematical explanations are given, and more precisely the conceptual resources available to the questioner in the analysis of the situation. The point raised by Sandborg is extremely relevant for our discussion, because the problem is not restricted to mathematical explanation within mathematics but affects also explanations of physical phenomena analysed through the lens of the why-questions techniques (Sandborg, 1998, p. 622).

Even if the idea of conceptual resources is left quite undeveloped in Sandborg's study, it is easy to identify such conceptual resources with the qualitative reinforcements proposed by some authors to Kitcher's model of explanation. Again, quantitative factors are not enough for the model to mirror the practice of scientist.

After this short summary of the models, it is now time to pass to a more concrete setting. In the following section I will present a case of MEPP coming from scientific practice.

3 Hénon-Heiles systems

Four decades ago, Michel Hénon and Carl Heiles were investigating the motion of stars about the galactic center. Rather than solve the problem with the actual potential of the galaxy (something which would have been quite difficult to achieve!), they restricted the motion to the xy plane, as in the Kepler problem, and studied a relatively simple analytic potential $U(q_x, q_y)$ that illustrates the general features of the problem (Hénon and Heiles, 1964). The Hénon-Heiles potential exhibits two cubic perturbation terms which couple together two standard harmonic oscillators:

$$U(q_x, q_y) = \frac{1}{2}(q_x^2 + q_y^2) + q_y q_x^2 - \frac{1}{3}q_y^3 \tag{1}$$

Accordingly, we call 'Hénon-Heiles systems' those systems formed by a particle moving in such a bidimensional potential.

Consider now the physical phenomenon under study, i.e. the motion of one particle moving in the Hénon-Heiles bidimensional potential $U(q_x, q_y)$, where the q_x and q_y are called 'generalized coordinates'. Take the motion of the system as our explanandum[4]. More precisely, we want to explain the behaviour (regular or not) of the system for different energies.

There are two mathematical ways to study the system. We can study the system through the Lagrangian analysis, or we can adopt the Hamiltonian formulation which comes with a particular mathematical structure called 'phase space'. The Lagrangian formulation is obtained introducing the Lagrangian function $L = T - U$, where T is the kinetic energy of the system, and successively obtaining the equations of the motion from the so called *Lagrange's equations*:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \tag{2}$$

In the Lagrangian formulation (nonrelativistic), a system with n degrees of freedom possesses n (second-order) differential equations of motion of the form (2). The state of the system is represented by a point in an n-dimensional *configuration space* whose coordinates are the n generalized coordinates q_i (q_x and q_y for the present bidimensional example). The motion of the system (as a function of time) can be interpreted as the path traced by this point as it traverses the configuration space. In the Lagrangian formulation, all the n coordinates must be independent. However, there is another formulation of the problem which is "based on a fundamentally different picture" (Goldstein, 2001, p. 335). In this formulation, called Hamiltonian formulation, we want to describe the motion in terms of *first-order* equations of motion. In order to do that, we double our set of independent quantities (thus obtaining $2n$ independent variables) by adding to our generalized coordinates q_i the new variables *conjugate* (or generalized) *momenta* p_i, defined as follows:

$$p_i = \frac{\partial L}{\partial \dot{q}_i}, \tag{3}$$

The quantities (q, p) are known as *canonical variables.*

From a mathematical point of view, the transition from Lagrangian to Hamiltonian formulation corresponds to changing the variables in our mechanical functions from $(q, \dot{q}, t)$ to (q, p, t), where p is related to q and $\dot{q}$ by equation (3). The procedure for switching variables in this manner, and the so called *Hamiltonian function* which is associated to it, is provided by a Legendre transformation:

$$H(q, p, t) = \sum_{k=1}^{n} p_i \dot{q}_i - L(q, \dot{q}, t) \tag{4}$$

If now we consider the differentials of the Lagrangian $L(q, \dot{q}, t)$ and of the Hamiltonian (4), we will obtain the $2n + 1$ relations

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \tag{5}$$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} \tag{6}$$

$$\frac{\partial L}{\partial t} = -\frac{\partial H}{\partial t} \tag{7}$$

The first two equations above are known as 'Hamilton's canonical equations of motion'. They are the desired set of first-order equations of motion which replace the n second-order Lagrange equations.

The space of the q and p coordinates is known as 'phase space'. Hence, the $2n$ canonical equations of the motion describe the behavior of the system point in the phase space, which has $2n$-dimensions and whose coordinates are the $2n$ independent variables. In other words, in the Hamiltonian formulation of mechanics the dynamics of the system is defined by the evolution of points ('trajectories') in this phase space.

For the case of our system, i.e. a particle moving in the bidimentional potential (1), the Hamiltonian function will be:

$$H = T + U = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}(q_x^2 + q_y^2) + q_y q_x^2 - \frac{1}{3}q_y^3 \tag{8}$$

And the respectively (nonlinear) equations of motion:

$$\frac{d^2 q_y}{dt^2} = \frac{dp_y}{dt} = \frac{\partial H}{\partial q_y} = -q_y - q_x^2 + q_y^2 \tag{9}$$

$$\frac{d^2 q_x}{dt^2} = \frac{dp_x}{dt} = \frac{\partial H}{\partial q_x} = -q_x - 2q_x q_y \tag{10}$$

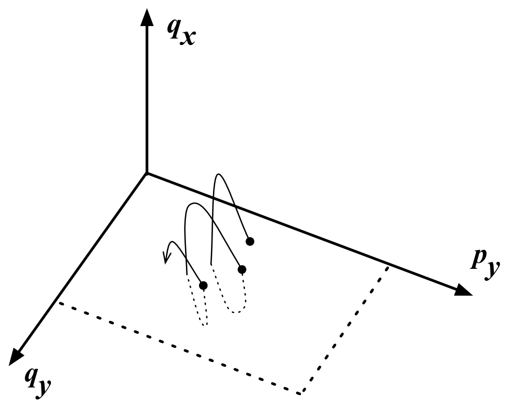

Fig. 1: The surface of section for the Hénon-Heiles problem is generated by recording and plotting the successive crossings of the $q_x = 0$ plane in the direction of increasing q_x.

These equations may be obtained from either Lagrange's equations or Hamilton's equations. Although both routes are admissible, however, scientists prefer the use of Hamiltonian formalism involving phase space theory. This is because, by using the Hamiltonian formalism, we can deduce visually, from a representation in the phase space, whether the system has a regular or chaotic motion[5]. In order to do that, we start by considering the total energy of the system E constant, thus lowering the dimensionality of the phase space by one:

$$E = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}(q_x^2 + q_y^2) + q_y q_x^2 - \frac{1}{3}q_y^3 \tag{11}$$

We take then a 2-dimensional cross section of this hypersurface in the phase space and we map the intersections of the trajectories with the plane by using a function called *Poincaré Map* (Figure 1). Finally, we look at the 'dots' made by the solutions (orbits) on the Poincaré section and we can

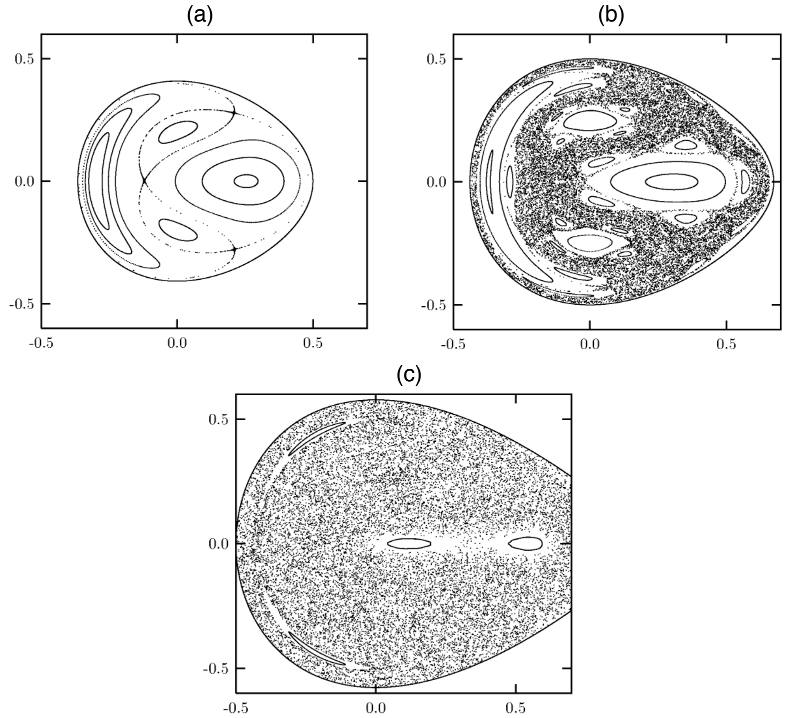

Fig. 2: Poincaré sections of the Hénon-Heiles system in the $q_y p_y$ plane ($q_x = 0$), showing several Henon-Heiles orbits. For $E = \frac{1}{12}$ orbits are regular (a); for $E = \frac{1}{8}$ we observe regions of regular motion and regions of chaos (b); finally, for energy $E = \frac{1}{6}$ chaos is dominant (c). Based on Hénon and Heiles (1964).

visually grasp qualitative informations about the dynamics of the system. We do that by following the order in which the dots appear. Solutions that never pass through the same arbitrarily small neighborhood of a point twice are chaotic (instead of following a regular curve, they are scattered and jump around in a more or less random fashion from one part of the Poincaré section to another). On the other hand, a dynamic state that gives rise to regular motion will have the property that nearby dynamic states will stay close to it as they get mapped around the plane. We use the mathematical 'resource' of Poincaré map because in the Hamiltonian formulation of our problem the phase space has $4N$-dimensions – points in the phase space are represented by quadruples of the form: (q_x, q_y, p_x, p_y) – and therefore the trajectories in this space (which define the dynamics of the system) are not directly visualisable on a diagram.

The sections for various energies summarize the dynamics at those energies. Studying the diagrams for different energies we can observe how at low energy the section is dominated by regular orbits (and then the associated motion is regular), at intermediate energy the section is divided more or less equally into regular and chaotic regions, while as we increase the total energy of the system the orbits become chaotic and the section is dominated by a single chaotic zone (Figure 2). Such transitions from regular to chaotic behavior are quite common; similar phenomena occur in widely different systems, though the details naturally depend on the system under study.

It is important to note, again, that the mathematical procedure involving phase space is not the only alternative for the study of the system. It is possible to analyse it via the Lagrangian formalism. However, as pointed out in a popular textbook on mechanics and dynamical systems:

> It is in such maps [Poincaré maps] that the fractal structure of chaotic dynamics becomes plausible [...] Therefore the use of phase space is inevitable as a means of understanding the structure accompanying chaos (Tamás and Márton, 2006, p. 22)

The hamiltonian formalism (including phase space and Poincaré map) is preferred by scientists because it allows to explain why the system behaves as such (in terms of energy). The importance of explaining the behavior of the system through this procedure is well recognized by scientists. On the other hand, the Lagrangian route seems not to carry the sense of explanatoriness that we obtain from the use of phase space theory. For instance, Lyon and Colyvan write "although there is a Lagrangian formulation of the theory in question that does not employ phase spaces, the cost of adopting such an approach is a loss of explanatory power" (Lyon and Colyvan, 2008, p. 2). Finally, observe that the possibility to reason visually on the diagram (thorugh Poincaré maps) has an essential role in the explanation provided.

It is now time to ask ourselves the following question: Are the two WTA accounts presented in Section 2 able to account for the mathematical explanation of Hénon-Heiles systems (MEPP recognized as such in scientific practice)?

4 Testing the Accounts

Kitcher's account is faced with an evident difficulty when used to account for cases of MEPP such as that concerning the behavior of Hénon-Heiles systems. The difficulty comes, as I am going to show, from the fact that the only way to explain the regular or chaotic behavior of the system is to appeal to the possibility to visualize the trajectories on the surface of section. According to scientists, this is an essential ingredient in the explanation provided. However, such inferential step (the inferential step in which we infere the regular or chaotic behaviour of the system by visualizing the trajectories) cannot be modelled by Kitcher's idea of argument pattern.

Recall that the explananda considered in Kitcher's unification model are members of K. Informally, we can think of K as the set of statements endorsed by an ideal scientific community at a specific moment in time. A statement in K, for instance that of the form 'Object O_1 has position P_1 and velocity v_1 at time t_1', is derived in Kitcher's model by using a particular argument pattern (for example, the Newtonian pattern). Furthermore, the same pattern is used to derive statements which do represent different phenomena (for instance, the Newtonian pattern is also used to derive the statement 'Object O_2 has position P_2 and velocity v_2 at time t_2', which refers to a physical phenomenon different from that considered by the previous statement having the same form). The unification model is then able (at least potentially) to tell that statements which represent different phenomena are derived from arguments that instantiate a *common* argument pattern[6].

In order to apply Kitcher's account for the present case of MEPP, two essential requisites should be fulfilled. First, the pattern of derivation used in the MEPP of the Hénon-Heiles system must represent an instance of a pattern used in deriving statements which concern the behavior of other physical phenomena as well. Otherwise, there would not be the unification idea that Kitcher assumes for explanation, and my testing would result trivial. Second, the statement concerning the behaviour of Hénon-Heiles system must belong to the set K of statements accepted by a scientific community at a particular time. For simplicity, let me indicate with the expression 'behaviour-statement' a statement concerning the behaviour of a physical system[7]. In our case, the behaviour-statement to consider is the following:

S_1 'The particle P has regular (or chaotic) behaviour at energy E'

Fortunately, both the requisites are met: the behaviour-statement S_1 belongs to K; second, there exist behaviour-statements concerning physical phenomena different from that related to S_1, and these statements are deduced via the same procedure used to derive S_1. With respect to the former, the behaviour-statement 'The particle P has regular (or chaotic) behaviour at energy E' belongs to K because it expresses a belief shared by scientists (it comes from our scientific practice, and it expresses a belief which is shared by the scientific community at a particular time). Furthermore, it is easy to see how the second requisite is satisfied as well. The very same procedure involving the Hamiltonian formalism, the phase space and the Poincaré Map, can be used to infer statements concerning the behaviour of physical phenomena different from the motion of a particle in a potential. For instance, the same procedure can be used in the case of the double pendulum, which is a different physical phenomenon[8].

If distinct behaviour-statements about the different phenomena are inferred through the very same procedure, according to Kitcher there should be an argument pattern which is used to derive these behavior-statements, thus providing unification. This suggests that the notion to be checked here is that of Kitcher's argument pattern.

To test Kitcher's notion of argument pattern for the present case amounts to answer the crucial question: is there a pattern (in Kitcher's sense) which is able to instantiate the Hénon-Heiles derivation and the derivation of the double pendulum? If yes, by considering the case of Hénon-Heiles system, an instantiation of this pattern would account for the particular derivation of the statement 'The particle P has regular (or chaotic) behaviour at energy

E'. In my testing below I am going to show that Kitcher's idea of argument pattern does not capture this particular derivation. More generally, I will point to the fact that Kitcher's argument pattern does not admit such types of derivations within its structure.

For Kitcher, a *general argument pattern* $\langle s, f, c\rangle$ is a triple consisting of a *schematic argument* s, a set f of filling instructions and a classification c for s. In the Newtonian case, the following schematic sentences (1)-(5) form a schematic argument s_N:

1. The force on α is β
2. The acceleration of α is γ
3. Force = mass $\cdot$ acceleration
4. (Mass of α)$\cdot$(γ) = β
5. $\delta = \theta$

The members of the set of filling instructions f_N are: 'all occurrences of α are to be replaced by an expression referring to the body under investigation'; 'occurrences of β are to be replaced by an algebraic expression referring to a function of the variable coordinates and of time'; 'γ is to be replaced by an expression which gives the acceleration of the body as a function of its coordinates and their time-derivatives'; 'δ is to be replaced by an expression referring to the variable coordinates of the body, and θ is to be replaced by an explicit function of time'. The set of filling instructions f_N contains the directions for replacing the dummy letters α, β, γ, δ, θ in every schematic sentence. The sentences contained in the classification set c_N for the schematic argument s_N give us the inferential information about the schematic argument: '(1)-(3) have the status of premisses'; '(4) is obtained from (1)-(3) by substituting identicals'; '(5) follows from (4) using algebraic manipulations and the techniques of the calculus'. Thus we have that a particular derivation in Newtonian mechanics, i.e. a sequence of sentences and formulas which accord with Newton's laws, instantiates the general argument pattern $\langle s_N, f_N, c_N \rangle$ just in case: (*i*) the derivation has the same number of terms as the schematic argument s_N, (*ii*) each sentence or formula in the derivation can be obtained from the corresponding schematic sentence in accordance with the filling instructions f_N, (*iii*) the terms of the derivation have the properties assigned by the classification c_N to members of the schematic argument s_N. The unifying power of Newton's theory consists in the fact that, by using the same Newtonian pattern of derivation $\langle s_N, f_N, c_N \rangle$ again and again, the theory shows us how to derive a large number of statements accepted by the scientific community.

Now, consider the laws of mechanics and the theory of differential equations as belonging to the corpus K of our beliefs. We want to construct an argument pattern (of the same kind as Kitcher's Newtonian pattern) which does instantiate the particular derivations which lead to the following behaviour-statements:

S_1 'The particle P has regular (or chaotic) behaviour at energy E'

S_2 'The double pendulum S has regular (or chaotic) behaviour at energy E'

Each statement above (S_1, S_2) is accepted in K, and each statement concerns a different physical phenomenon (respectively, the motion of a particle in a potential and the motion of a double pendulum). The behaviour-statement S_1 and S_2 are both obtained by finding the equations of motion, constructing the Poincaré section with the relative Map, and finally grasping visually the behaviour of the system for fixed energies. Observe that there are plenty of behaviour-statements (concerning different physical phenomena) that can be obtained through the same procedure. For instance, statements about the regular or chaotic behaviour of the voltage in a triode circuit (modelled by Van der Pol oscillator), about the behaviour of a spring pendulum whose spring's stiffness does not exactly obey Hooke's law (modelled by the Duffing oscillator), or even about the behaviour of the simple pendulum or the undamped spring-mass system (the latters modelled by a simple harmonic oscillator)[9]. The list of statements above contained only two of them. Now, if the steps in the derivation which involve the equations of motion and the construction of the Poincaré section can be mirrored by an argument pattern *à la* Kitcher, the inferential step which appeals to the possibility of visualizing the trajectories cannot. This is evident if we look at the Newtonian pattern presented above. How would such a (visual) inferential step appear in the schematic argument s? And what kind of filling instructions and classification would be able to capture it? The derivation which is performed in the Newtonian case can be mirrored by a formal deductive schema of the kind Kitcher proposes, but the derivation used in the MEPP of Hénon-Heiles system does appeal to an ingredient which cannot be mirrored by the idea of argument pattern. Moreover, this ingrendient is recognized as essential by scientists and therefore it is reasonable to include it in the pattern-structure. If there exits a common pattern for behaviour-statements like S_1 and S_2, then, it seems that this pattern has a structure essentially different from that of the Newtonian pattern $\langle s_N, f_N, c_N \rangle$ proposed by Kitcher.

To sum up, an argument pattern *à la* Kitcher does not admit inside within its structure a particular inferential step recognized by scientists as essential to the explanation provided. It is then inappropriate to use it for derivations of behaviour-statements such as that concerning the Hénon-Heiles system, or for similar derivations of behaviour-statements concerning different phenomena. Hence, Kicther's unification model, at least in its original form, is not able to account for the MEPP concerning the behaviour of the Hénon-Heiles system and is in conflict with the intuitions of the scientists[10].

Consider now Van Fraassen's model, which treats explanations as answers to why-questions. Our why-question is 'Why Hénon-Heiles system has chaotic behaviour at energy E?', where the topic P_k is 'Hénon-Heiles system has chaotic behaviour at energy E'. In the contrast class X we have, together with the topic, the alternative proposition 'Hénon-Heiles system has regular behaviour at energy E'. The answer to the why question is given by 'Because A', where A is the proposition 'Solutions –the dots– are scattered on Poincaré section. They never pass through the same arbitrarily small neighborhood of a point twice'.

Now, recall that according to Van Fraassen, B is a *direct answer* (an explanation) to our why-question $Q = \langle P_k, X, R \rangle$ exactly if there is some proposition A such that A bears a relation R to $\langle P_k, X \rangle$ and B is the proposition which is true exactly if: the topic P_k is true; only the topic is true in the constrast class X (formed only by the two propositions 'Hénon-Heiles system has chaotic behaviour at energy E' and 'Hénon-Heiles system has regular behaviour at energy E') ; and A is true. There are at least two problems with this approach as applied to our example, namely: the problem of determining the relevance relation R and the problem of what a why-question presupposes. Both point to the fact that the explanation considered cannot be captured by a why-question analysis. I will address here only the second problem (but this is sufficient to reject Van Fraassen's model as suitable to cover the present case).

Consider the last part of Van Fraassen's definition of direct answer: B is a *direct answer* to Q if B is the proposition which is true exactly if: P_k is true; only P_k is true in X; and A is true. Moreover, keep in mind that we want Van Fraassen's account to agree with scientific practice in considering the case discussed as a genuine explanation. The difficulty for Van Fraassen's approach is noticeable. To say that only P_k is true in X (and then to regard the proposition 'Hénon-Heiles system has regular behaviour at energy E' as false) is grounded exactly in the possibility to qualitatively grasp the behaviour of the system via Poincaré map and to claim that proposition A is true. In other words, we know that the Hénon-Heiles system has cahotic behaviour at energy E because we use Poincaré map and we infer visually that solutions never pass through the same arbitrarily small neighborhood of a point twice. But this means that, according to a why-question analysis, the explanatory activity admitted assumes the form of a display of consequences (the topic P_k) of what we have already accepted as given (the proposition A). Very roughly, this would amount to say: 'this is an explanation because we have already accepted that it is an explanation'. This is, in fact, the general moral of Sandborg's criticism: "The key point is that a why question is taken to implicitly fix the way an answer must regard its topic" (Sandborg, 1998, p. 621).

It is clear then that either the why-question approach is not able to account for our MEPP[11]. And therefore both the two WTA models discussed above are not able to capture a mathematical explanation of physical phenomena recognized as such in scientific practice, and consequently should be refined or abandoned.

5 Conclusion

A general conclusion from the criticisms presented above (subsections 2.1 and 2.2) is that we need to introduce a qualitative component into our model of MEPP in order to account for the intuitons coming from scientific practice. This suggests that there is a link between explanation and qualitative (rather than purely quantitative) factors in the practice of scientists who explain phenomena in science. It can be conjectured that an example of such qualitative component does appear in the case of Hénon-Heiles systems, and that it is given by a particular form of reasoning (visual reasoning) through which we can look at the diagram and infer informations about the behaviour of the system. Despite having an essential role in the explanation provided by the scientists, however, this ingredient is not captured by the two models, as my assessment shows.

It can be noted that MEPP do not always involve visual reasoning and it is recognized that there are different MEPP that use other specific forms of reasoning as well. For instance, in his book *The Devil in the Details*

(2002), Robert Batterman has argued that particular species of MEPP, called 'asymptotic explanations', gain their explanatory power by the systematic throwing away of various causal and physical details. By using asymptotic methods, i.e. methods which "eliminate detail and precision", the scientist is able to obtain a particular mathematical explanation for a phenomenon (or class of phenomena). The reasoning which is generated from asymptotic methods is called by Batterman 'asymptotic reasoning', and 'asymptotic explanation' is exactly the kind of explanation which utilizes such specific kind of reasoning. The fact that unification models and other accounts of explanation do not manage to do a good job for such MEPP was one of Batterman's motivations for introducing his own perspective (Batterman, 2002, p. 35).

The main example discussed by Batterman concerns the explanation offered in condensed matter physics for the universality of critical phenomena. The mathematical technique of Renormalization Group Theory (RGT) is what permits to reason asymptotically and obtain an explanation for the universality of critical phenomena. Note that Batterman's intuitions about the role played by such particular kind of reasoning is strongly supported by examples from scientific practice. Kenneth Wilson, the high energy theorist who formulated the RGT in 1971, in his Nobel lecture "The Renormalization Group and Critical Phenomena" (8 December 1982) gave particular emphasis to the crucial role played by RGT in the explanation of the universal behaviour of different systems. Illustrating the RGT approach to critical phenomena, he says: "This [renormalization group analysis] leads to an explanation of the universality of critical behavior for different kinds of systems at the atomic level" (Wilson, 1982).

Another form of reasoning which is recognized to come as essential ingredient in our explanatory scientific practices is analogical reasoning. Analogical reasoning is the process of reasoning by analogy, i.e. reason and learn about a new situation (the 'target' analog) by relating it to a more familiar situation (the 'source' analog) that can be viewed as structurally parallel (Holyoak and Thagard, 1997). While this particular kind of reasoning is used extensively in our everyday-life, there is a number of philosophers who welcome the idea that the use of analogical reasoning in science does provide an essential contribute to scientific explanation (Hesse, 1966).

Far from giving a bestiary of the kinds of reasoning we find associated to MEPP in scientific practice, the purpose of the previous paragraphs was to suggest that MEPP involve specific forms of reasonings. But these forms of reasoning are essentially distinct (for instance, analogical reasoning is distinct from visual reasoning) and therefore it might be thought that they characterize different 'species' of MEPP. To accept this new perspective has an immediate consequence on our methodology and on the possibility to study MEPP by proposing a WTA model. In order to account for this variety, or species, of explanations we have to accept that pluralism is the best attitude to adopt (at least if we take the intuitions of scientists seriously, which is what I assume as a basic premise of my investigation). It is very hard, in fact, to see how this picture of MEPP can fit within the traditional WTA view[12].

If my testing above is correct, and if scientific practice suggests that specific forms of reasoning are a 'mark' of MEPP, how could the WTA partisan argue against the need of adopting a pluralist perspective? There are, I think, three possible ways to reject pluralism as the better attitude to adopt toward MEPP. A first strategy would consist in denying that such forms of reasoning play an essential role in MEPP or that examples such as that of Hénon-Heiles system do represent genuine cases of MEPP. But, again, this option is not the option our contemporary science seems to suggest. If we want to take the work of scientists seriously, we have to accept the test-case considered as genuine case of MEPP and follow the scientists in looking at the form of reasoning which is involved as a crucial explanatory ingredient. A second move would be to propose a new encompassing WTA model, i.e. a model able to account for all the varieties of MEPP (among them the case of Hénon Heiles systems). However, at the best of my knowledge, we do not dispose of such a model. Finally, it might be that one of the WTA models considered up to now, or possibly both, could be refined in order to capture the specific forms of reasoning occurring in MEPP and reflect the intuitions of the scientists. But this obviously shifts the burden of the proof to the partisans of the WTA approach.

To sum up, I did not provide any a priori account of MEPP but I focused on a MEPP recognized as such in scientific practice. I evaluated on this test case two theories which are among the leading contemporary WTA accounts of scientific explanation, those of Kitcher and Van Fraassen, and I showed that they have difficulties in accounting for the explanatory character of this case. I pointed out that my evaluation and independent considerations coming from the literature reveal a different picture of MEPP which is in need to be explored, and it is to this new picture that I turned my attention. This new perspective, which is only sketched here, is based on a paradigm very different from that which stands behind traditional theories of scientific explanation such as Kitcher's or Van Fraassen's. It shifts the attention of the MEPP-scholar to the particular forms of reasoning which are employed in MEPP and which are recognized by scientists as essential in their explanatory practices. Furthermore, this perspective looks at pluralism as the driving force in the investigation of MEPP.

Again, what I have offered here is only a vague idea of how the study of MEPP can be approached once we adopt this view. And, unfortunately, at this stage I do not have very much to say on how a particular form of reasoning can be captured through a philosophical notion, or even how it does provide explanatoriness and contributes to the explanation. My modest considerations above are based on an observation of scientific practice, and this observation points to the evidence that such forms of reasoning are crucial to MEPP. Let me note, however, that such a perspective has not been put forward in the literature on MEPP before. Optimistically, its examination may come out as fruitful and open different directions of analysis, thus contributing to a debate which is only at its earliest stage and whose development could have strong repercussions on different areas of philosophy of mathematics and general philosophy of science.

Notes

[1]The fact that Kitcher's theory cannot account for a particular case of MEPP has been pointed out by Batterman (2002, p. 35). Let me note, however, that Batterman provides only a general discussion and not a detailed analysis.

[2]The only formal constraint on the relevance relation is that it obtains between a

proposition A and topic/contrast-class pairs $\langle P_k, X\rangle$. Van Fraassen does not offer any other relevance requirements on R in its formal characterization. I will return to this point, central to Philip Kitcher and Wesley Salmon's criticism of Van Fraassen model, below.

[3]Consider, for instance, the following quotation from Matti Sintonen: "It also becomes obvious that not all explanations are answers to why-questions. Depending on the type of inquiry at hand they could be how-questions, how possible-questions, what-questions, or the like" (Sintonen, 1999, p. 134).

[4]The example of Hénon-Heiles system has been used by Lyon and Colyvan in their paper *The Explanatory Power of Phase Spaces* (Lyon and Colyvan, 2008). However, they discussed the case in the context of the nominalist-platonist debate in philosophy of mathematics. My analysis here points to aspects which are quite far from Lyon and Colyvan ontological considerations. I will not concentrate on the role that MEPP like this are supposed to play in the ontological dispute, but rather on the fact that this case is considered as a genuine explanation by scientists. This marks an essential difference in the analysis and in the direction of investigations which follows.

[5]Chaos is a motion which is, simultaneously: (a) irregular in time (it is not simply the superposition of periodic motions, it is really aperiodic); (b) unpredictable in the long term and sensitive to initial conditions; (c) complex, but ordered, in the phase space (it is associated with a fractal structure) (Tamás and Márton, 2006, p. 22).

[6]Consider, for instance, that O_1 is a ball and O_2 is a satellite. The orbiting of the satellite around the Earth and the falling of the ball from a tower are not the same phenomena. However, according to Kitcher, they are covered (and unified) by the same Newtonian pattern, i.e. the arguments from which we derive the two statements 'Object O_1 has position P_1 and velocity v_1 at time t_1' and 'Object O_2 has position P_2 and velocity v_2 at time t_2' do instantiate the same argument pattern.

[7]Intuitively, statements such as 'Mark played his new guitar during the concert' are not among the behaviour-statements I am considering here.

[8]In the same way of the Hénon-Heiles system, the regular or chaotic behaviour of this system can be established by using the Hamiltonian formalism and then looking at the

trajectories made on the surface of section. See Chapter 5 of (Korsch et al., 2008) for a study of the double pendulum.

[9]In the latter unidimensional cases (simple pendulum and undamped spring-mass system) the surface of section coincides with the whole phase space. To study the trajectories on the surface of section is the same as to study the trajectories in the phase space.

[10]Observe that the fact that Kitcher's argument pattern does not reflect the kind of inferences made in the Hénon-Heiles example is sufficient to show the inapplicability of Kitcher's account for the present case. Further considerations about the number of conclusions generated by the pattern are not necessary once the basic idea of argument pattern comes as inapplicable to our case.

[11]The failure of the *why*-question approach for cases of MEPP such as the case I propose here can be attributed to the fact that those cases do not come under the form of *why*-questions. As I showed above, to subsume our explanandum in a *why* question amounts to fix 'a priori' the way the answer regards the topic. Perhaps in cases such as that of the Hénon-Heiles system example, a more promising direction of analysis would be of adopting a *What*-question approach. For instance, re-formulating the explanatory question as '*What* is the behaviour of the system at Energy E?'. As in commonly spoken language, explanations in science are associated not only with with informative answers which are responses to why-questions, but also with *what* or even *how*-questions (Faye, 1999; Sintonen, 1999).

[12]Perhaps another point which supports the advantage of the pluralist perspective comes from the fact that by adopting it we can account for a greater number of cases of MEPP (cases of MEPP recognized as such in scientific practice) which otherwise would be excluded from our philosophical investigation. This constitutes, I think, a quite important step ahead in the study of MEPP.

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